

Forward Problems and their Inverse Solutions

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Outline

1 Forward Problem Example

- Weather Forecasting

2 Inverse Problem Example

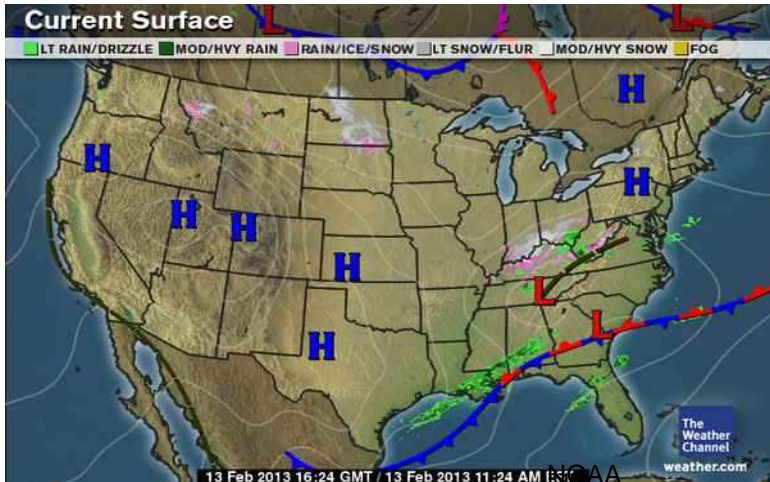
- CT Scan

3 Line Fitting

- Least Squares
- Direct Method
- Sampling Method

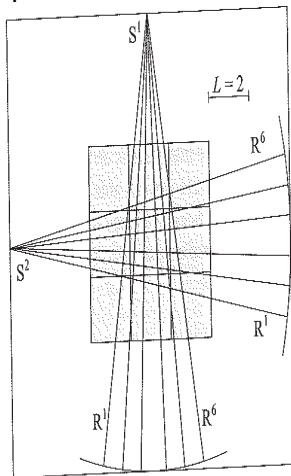
4 Summary and Future Work

Weather Forecasting

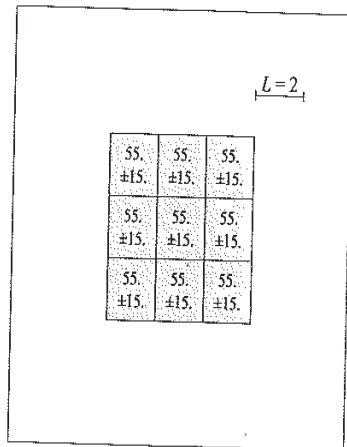


CT SCAN (1/4)

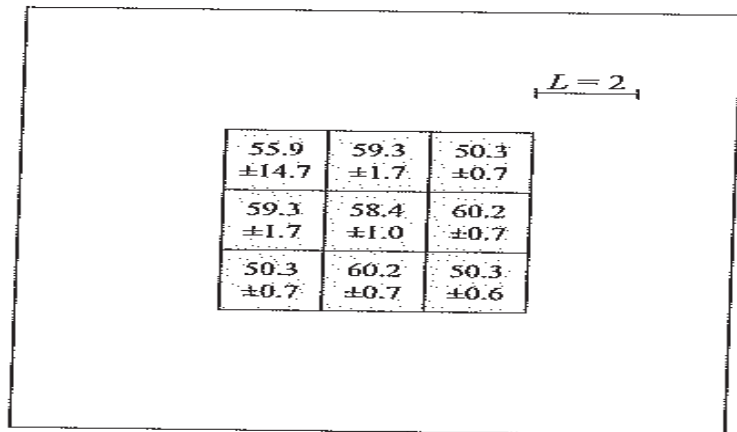
Setup:



InitialGuess:



CT SCAN (2/4)

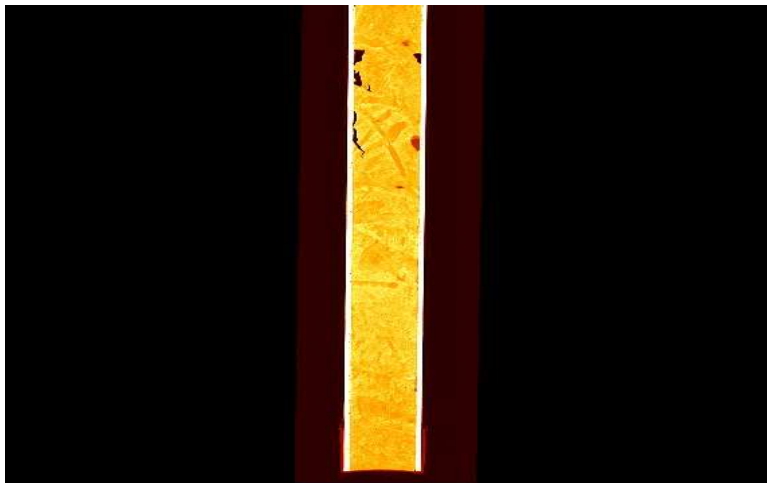


Source: Tarantola, 2005

CT SCAN (3/4)



CT SCAN (4/4)

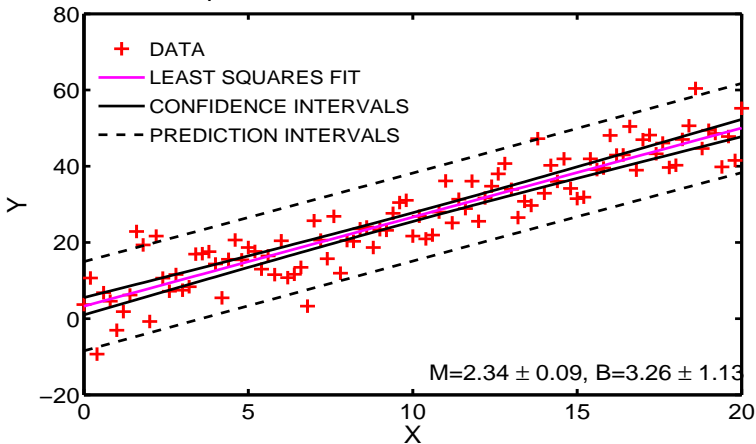


"Simple" Case of Line Fit

- A simple inverse problem is " $y=mx$ ". Instead of knowing m and x to solve for y , you solve for either x or m given knowledge of y and m or y and x .
- What we can easily measure is often not what we want to know.
- In an inverse problem, we use observations to infer what we want to know.

Least Squares

Least Squares Fit to Data with Confidence Intervals



Bayesian Methods

- Part 1: Direct Method
- Part 2: Sampling Method (MCMC)

Bayesian vs. Least Squares

- start with initial guess
- solve for a distribution on the parameters (many solutions)

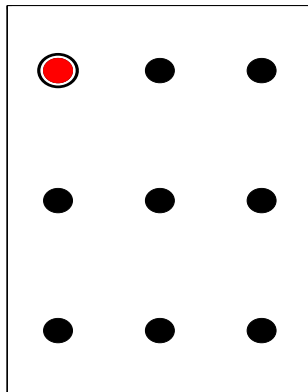
Bayesian

- initial guess of parameters is called the prior distribution (shapes solution)
- "I know almost nothing about the parameters" → assume uniformly distributed

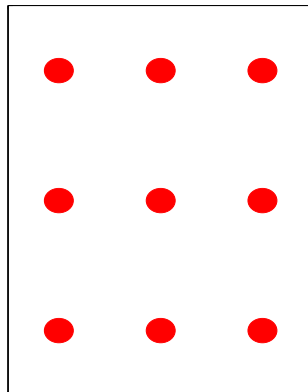
Part 1: DIRECT METHOD

Direct Method

Number of Slope Values: M



Number of Intercept Values: B

Number of Calculations: $M \times B = 81$

Direct Method

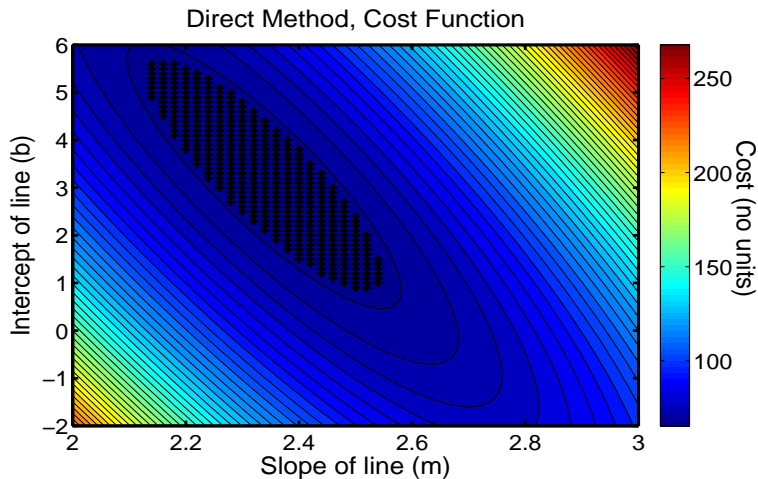
Finding the best solutions:

- Define a cost function:

$$2J(m, b) = (g(m) - d)^T C^{-1} (g(m) - d) = \sum_1^N \frac{((mx + b) - y)^2}{\sigma^2} \quad (1)$$

- best solutions are where cost is within 2σ of the mean (95% confidence interval)

Direct Method



Direct Method

Finding the best solutions:

- Define a cost function:

$$2J(m, b) = (g(m) - d)^T C^{-1} (g(m) - d) = \sum_1^N \frac{((mx + b) - y)^2}{\sigma^2} \quad (2)$$

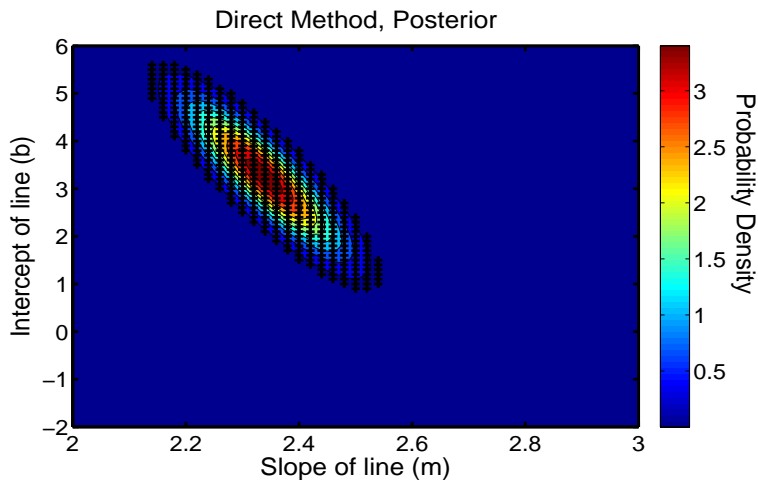
- best solutions are where cost is within 2σ of the mean (95% confidence interval)



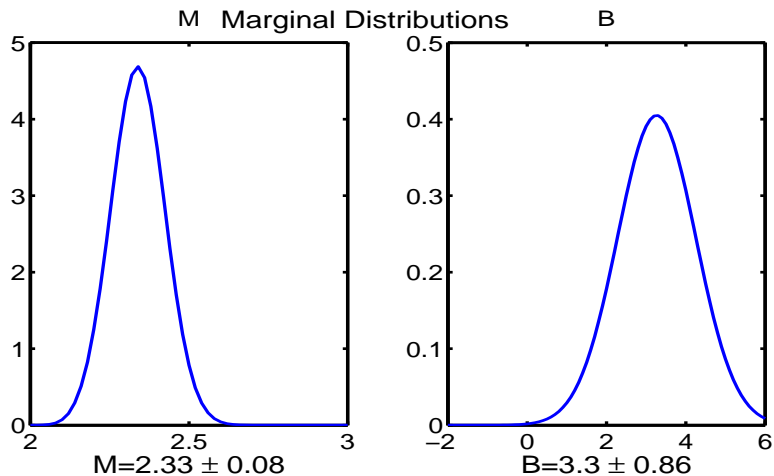
$$P(m, b) = e^{-J(m, b)} \quad (3)$$

is the posterior distribution on the parameters (central limit theorem)

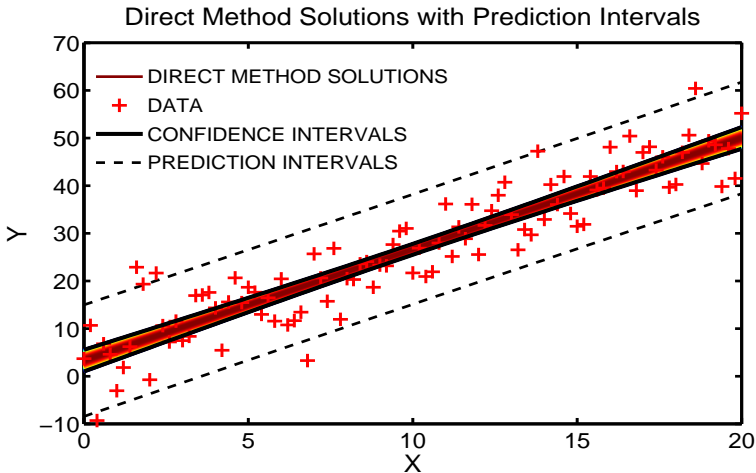
Direct Method



Direct Method

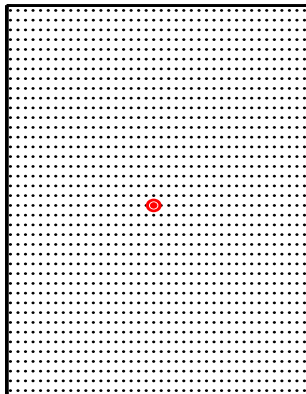


Direct Method

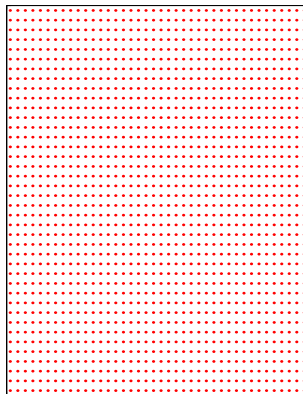


Direct Method

Number of Slope Values: M



Number of Intercept Values: B



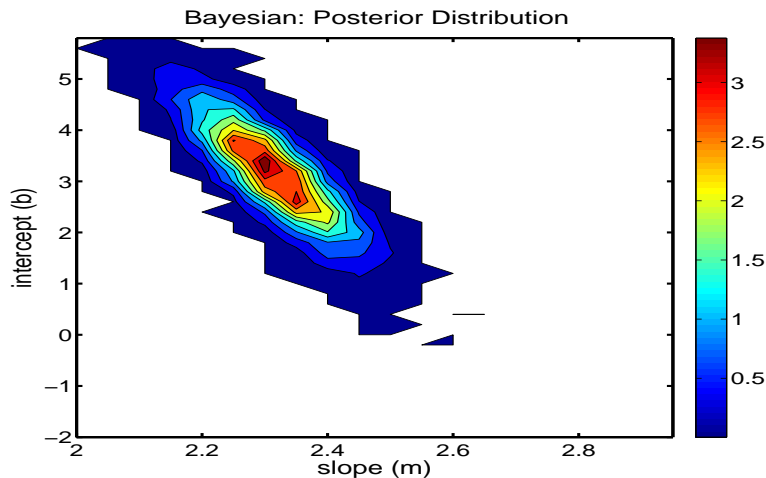
Number of Calculations: $M \times B = 2.56 \times 10^6$

Part 2: Sampling Method (MCMC)

Metropolis-Hastings: Monte Carlo Markov Chain Algorithm

- **Step 1: Start with (m_0, b_0)**
- **Step 2: Step a random distance to (m_1, b_1)**
- **Step 3: If $P(m_1, b_1) < P(m_0, b_0)$ accept (m_1, b_1) .**
- **Step 4: If not, calculate a transition probability $P(m_1, b_1)/P(m_0, b_0)$. Calculate a random number ϵ between 0 and 1. If $P(m_1, b_1)/P(m_0, b_0) \geq \epsilon$, accept (m_1, b_1) . Otherwise, reject.**
- **Step 5: If (m_1, b_1) is accepted, start Step 2 at (m_1, b_1) . Otherwise, start Step 2 at (m_0, b_0)**

Metropolis-Hastings

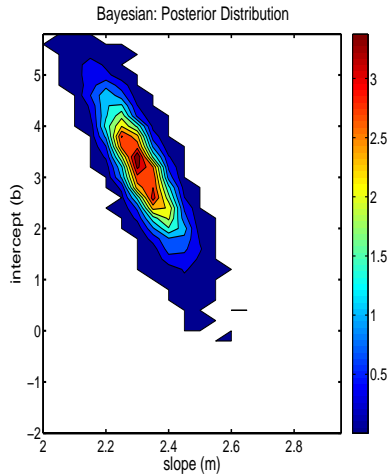
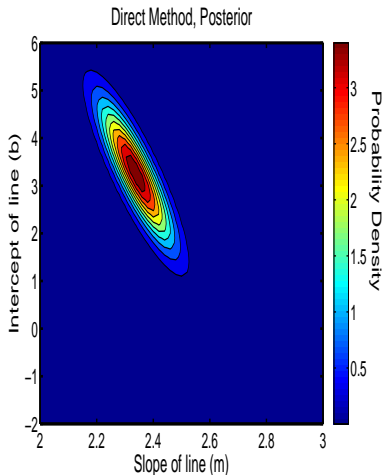


A histogram showing the frequency of the number of children per family. The x-axis represents the number of children (2, 2.5, 3) and the y-axis represents the frequency (0 to 3500). The distribution is unimodal and slightly right-skewed, peaking at approximately 3200 for 2.4 children.

A histogram showing the frequency of iterations for 1000 trials. The x-axis is labeled 'Iterations' and ranges from -2 to 6. The y-axis is labeled 'Frequency' and ranges from 0 to 3000. The distribution is unimodal and slightly right-skewed, peaking at approximately 2800 iterations.

$$B = 3.3 \pm 0.94$$

Metropolis-Hastings



Summary

- Least squares: maximum likelihood solution with an error bar.
- Direct methods: could miss mass in posterior distributions, computationally expensive
- Sampler methods: don't cover all parameter space (can miss multiple peaks in cost function), easier on computer

Research Applications

Current work:

- estimation of wind stress under hurricane forcing conditions from consideration of observed ocean response
- constraining constant parameters in turbulent mixing models to observations of ocean state in tropical Pacific

Future research:

- sampler used (Jackson et al., 2004)
- definition of the cost function (Mu et al., 2004)
- development of solution smoothers

Acknowledgements

Thanks to Deborah Khider and Charles Jackson for help with all aspects of putting this talk together.